Tapered Cables

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Vertical Space Elevator Cables

Space elevator cables using known materials must be tapered − thicker at the middle, thin at the ends. An equatorial space elevator cable starts at an altitude of 6378 kilometers above the center of the Earth, which is rotating at 72.9 microradians per second. Thus, equatorial rotational velocity is 465 meters per second at sea level. The gravitational acceleration is 9.81 meters per second, with 0.03 meters per second subtracted for centrifugal acceleration.

A straight cable extending to infinity from the equatorial surface, and rotating with the earth at angular frequency ω, will (at radius r) see a centripedal acceleration of:

$$a(r) = a_g (R_o / r)^2 - \omega^2 r \tag{1}$$

The first term is the gravitational acceleration, equal to the gravitational acceleration a_g when the radius r equals the radius of the Earth's surface, R_0. The second term is the centrifugal force, which is the angular frequency ω squared times the radius r. The first term diminishes with r, while the second term increases.

Geosynchronous orbit is the radius where this acceleration is zero, or

$$r_{GEO} = \left(a_g \left(\frac{R_0}{\omega} \right)^2 \right)^{\frac{1}{3}} \tag{2}$$

Which computes to 42,200 kilometers radius, or 35,800 kilometers altitude. A cable segment at radius r with a length dr and a weight per unit length of $m_r = \rho A(r) dr$ will add a tensile force $df(r)$ to the cable of

$$df(r) = \rho A(r)a(r)dr = \rho A(r)\left(a_g(R_o / r)^2 - \omega^2 r\right)dr \tag{3}$$

The force in the cable should be some fraction D of the maximum sustainable force that the cross section $A(r)$ can stand. D should be less than 0.7 or so. The cable has to support non−structural elements, and will also have axial tension vibrations due to uneven acceleration or jerks from the payload drive motors. It will take quite a bit of engineering magic to make D as large as 0.7; smaller numbers may be more prudent. So the force in the cable is designed to be:

$$f(r) = DSA(r) \tag{4}$$

Combining (3) and (4) we get the following differential equation for the cable cross section $A(r)$:

$$DSdA(r) = \rho A(r)\left(a_g(R_o / r)^2 - \omega^2 r\right)dr \tag{5}$$

This can be recast:

$$D\left(\frac{S}{\rho a_g}\right)\frac{dA(r)}{A(r)} = \left(\left(\frac{R_o}{r}\right)^2 - \frac{\omega^2}{a_g} r\right)dr \tag{6}$$

and integrated:

$$D\left(\frac{S}{\rho a_g}\right)\int_{A_0}^{A}\frac{dA(r)}{A(r)} = \int_{R_o}^{r}\left(\left(\frac{R_o}{r}\right)^2 - \frac{\omega^2}{a_g} r\right)dr \tag{7}$$

Defining $A_0 \equiv A(R_0)$ as the cross section at the Earth's surface, (7) can be solved as

$$\ln\left(A(r) / A_0\right) = \left(\frac{R_o}{DL_{SUPPORT}}\right)\left(\left(1 - \frac{R_0}{r}\right) + \left(\frac{\omega^2 R_0}{2a_g}\right)\left(1 - \left(\frac{r}{R_0}\right)^2\right)\right) \tag{8}$$

where the support length $L_{SUPPOET} \equiv \frac{S}{a_g \rho}$ is the strength to weight ratio.

In exponential form:

$$A(r) = A_0 \exp\left(\left(\frac{R_o}{DL_{SUPPORT}}\right)\left(\left(1 - \frac{R_0}{r}\right) + \left(\frac{\omega^2 R_0}{2a_g}\right)\left(1 - \left(\frac{r}{R_0}\right)^2\right)\right)\right) \tag{9}$$

This equation grows rapidly slightly above the surface, where r is only slightly larger than R_0, then grows slowly, reaching a maximum at geosynchronous radius r_{GEO}:

$$A_{MAX} = A(r_{GEO}) = A_0 \exp\left(\left(\frac{R_0}{DL_{SUPPORT}}\right)\left(1 - \frac{3K}{2} + \frac{K^3}{2}\right)\right)$$

(10)

where the dimensionless factor K is defined as:

$$K \equiv \frac{R_0}{r_{GEO}} = \left(\frac{\omega^2 R_0}{a_g}\right)^{\frac{1}{3}} \approx 0.151 \tag{11}$$

This is small; equation (10) can be approximated as:

$$A_{MAX} \approx A_0 \exp\left(\frac{0.775 \times R_0}{DL_{SUPPORT}}\right) \tag{12}$$

As you can see from (12), if the support length is much shorter than the Earth's radius, the exponential can grow very large.

The volume of the cable can be integrated out to infinity. Obviously, a real cable will be terminated with a lump mass somewhere above geosynchronous orbit. However, more terminating force is possible with a smaller total mass if an infinite cable is assumed. Since the infinite case is easier to calculate, we will integrate (9), modified by (11), to infinity:

$$Volume = A_0 \int_{R_0}^{\infty} \exp\left(\left(\frac{R_o}{DL_{SUPPORT}}\right)\left(\left(1-\frac{R_0}{r}\right)+\left(\frac{\omega^2 R_0}{2a_g}\right)\left(1-\left(\frac{r}{R_0}\right)^2\right)\right)\right)dr \qquad (13)$$

Lets make two further substitutions, $X \equiv \frac{R_o}{DL_{SUPPORT}}$ and $y \equiv \frac{r}{R_0}$, transforming (13) into:

$$Volume = A_0 R_0 \int_{1}^{\infty} \exp\left(X\left(1+\frac{K^3}{2}-\frac{1}{y}-\frac{K^3}{2}y^2\right)\right)dy$$

(14)

I don't know how to integrate that analytically; so, let's integrate it numerically for various values of X. We get the following table (D=0.7) :

Material	$L_{SUPPORT}$	X	$\frac{A_{MAX}}{A_0}$	$\frac{Volume}{A_0 R_0}$
Kevlar 49 & Epoxy	84 km	108.5	3.30e36	8.04e36
Zylon PBO & Epoxy	200 km	45.6	2.23e15	8.19e15
Nanotech Quartz	665 km	13.7	4.09e4	2.68e5
Carbon Nanotube	2,500 km	3.6	16.3	201
Nanotech Diamond	15,000 km	0.6	1.59	62

Note: the last three numbers are not experimentally demonstrated, nor have any of these numbers been peer reviewed − use at your own risk!

Obviously, a practical space−elevator cable will require nanotechnology or something much like it. Assuming diamond materials launched into orbit and a space elevator constructed from space, it will take a while for a space elevator to "pay back" its own mass.

The most effective space elevator would accelerate small payloads quickly to launch velocities, so they can free−fall up the cable, relieving strain as rapidly as possible. Assume a steady stream of small payloads with a surface rest weight of 1/3 of that supportable by the bottom of the cable, and that the payloads are launching at 3 gees, for a net vertical acceleration of 2 gees. These payloads will reach geosync−capable velocities of ≈0.88 escape velocity, or 9850 m/s^2, in 500 seconds, at which point another payload may be lifted. If each payload contains nothing but more cable stuff, the payload volume will be $L_{SUPPORT} \times A_0 / 3$. A carbon nanotube cable could thus lift the mass of a second nanotube cable in about 770Kseconds (absolute best case!) while a nanotech diamond cable could lift its mass in 40K seconds. However, neither of these best−case scenarios include propulsion motors for the 3 gee boost; smaller boost rates and extra mass increase the numbers significantly.

If a diamondoid vertical space elevator cable is 5 mm in diameter at the ground, it will have a total volume of 7700 cubic meters, and a mass of 10,000 tons. Lifting this amount of mass to geosyncronous orbit requires 4.8e14 joules, or 12 GW of power at the 40Ksec rate. Obviously, this will take cables much larger than 5mm in diameter to carry, and extremely large motors in the payloads − the calculation breaks down. The actual limit on space elevator replenishment rate will probably be set by other considerations than "perfect lifting capacity".

Diagonal Tapered Cables:

The Launch Loop uses much shorter cables, but they do not hang straight down. This complicates the analysis, but on the other hand we can safely make the worst case assumption that the gravity field does not change significantly over the height of the Loop (< 4%).

The diagonal cables are intended to relieve vertical, lateral, and axial forces. This requires three cables per set, and horizontal forces in each set. We will examine each cable separately, using x and y coordinates and l as the dependent length coordinate. The force is axial to the curving cable, with the x component f_x constant. The f_y force component increases vertically with the gravitational weight of the cable. We will use a safety factor of D and a support length of $L_{SUPPORT}$ as before.

The cable starts at the surface *y=0* with a slope y'_0. Each incremental segment of cable *dy* has a slope y', a length $dl = dy\sqrt{1+1/y'^2}$, an axial force

of $f = f_y\sqrt{1+1/y'^2}$, a weight proportional mass proportional to length and cross section proportional to force of $df_y = \rho a_g A dl = f_y\left(\frac{1+1/y'^2}{DL_{SUPPORT}}\right)dy$. This is fairly easy to integrate, resulting in

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